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NUMERICAL IDENTIFICATION OF BOUNDARY  
CONDITIONS ON NONLINEARLY RADIATING  
INVERSE HEAT CONDUCTION PROBLEMS\*

by

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ABSTRACT

An explicit and unconditionally stable finite difference method for the solution of the transient inverse heat conduction problem in a semi-infinite or finite slab mediums subject to nonlinear radiation boundary conditions is presented. After measuring two interior temperature histories, the mollification method is used to determine the surface transient heat source if the energy radiation law is known. Alternatively, if the active surface is heated by a source at a rate proportional to a given function, the nonlinear surface radiation law is then recovered as a function of the interface temperature when the problem is feasible. Two typical examples corresponding to Newton cooling law and Stefan-Boltzmann radiation law respectively are illustrated. In all cases, the method predicts the surface conditions with an accuracy suitable for many practical purposes.

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## 1. Introduction.

In this paper we investigate the numerical identification of surface transient heat sources in one-dimensional semi-infinite and finite slab mediums when the active surface radiates energy according to a known nonlinear law. Alternatively, if the active surface is heated by a source at a rate proportional to a given function, the nonlinear radiating boundary condition is then numerically identified as a function of the interface temperature if the problem is feasible.

These two tasks can be viewed as suitable generalizations of the classical problem of attempting to determine the interface temperature between a gas and a solid with a nonlinear heat transfer law. The existence and uniqueness of a strictly increasing solution of the semi-infinite body version of this problem has been considered by Mann and Wolf [Ref.7] for a monotone Lipschitz radiation law. Roberts and Mann [Ref.10] extended the previous result after removing the Lipschitz condition on the nonlinear heat transfer law. Keller and Olmstead [Ref.6] investigated the same problem in the presence of a positive integrable transient source and introduced a constructive proof for existence and uniqueness of the interface temperature by the method of lower and upper solutions. The numerical solution of the nonlinear Volterra integral equation characterizing the active surface temperature history was implemented by Chambré [Ref.1] using the method of successive approximations and, more recently, by Groetsch [Ref.3] who successfully combined Abel inversion formula with B-spline approximation and product integration. A natural extension of this technique to solve the same problem in the finite slab medium is discussed in Groetsch [Ref.4]. Also for the finite slab case, Villaseñor and Squire [Ref.12] have proposed a numerical procedure based on a generalized trapezoidal rule and Richardson extrapolation. More general problems of the same kind, combining the effects of convection and radiation at the interface, can be found in Friedman [Ref.2] and Saljnikov and Petrovic [Ref.11].

In all the works mentioned above, the nonlinear radiation law and the transient boundary source are supposed to be known in order to determine the interface temperature. Consequently, if the new task consists on the identification of the nonlinear radiation law or on the identification of the transient boundary source function, a different approach must be used.

It is possible to estimate the surface temperature and the surface heat flux in a body from measured temperature histories at fixed locations inside the body. However, this Inverse Heat Conduction Problem (IHCP) is an ill-posed problem because small errors in the data induce large errors in the computed surface heat flux history or in the computed temperature history solutions and, consequently, special methods are needed in order to restore continuity with respect to the data. In this paper we consider initially, the solution of a one-dimensional IHCP by a fully explicit and stable space marching finite difference implementation of the Mollification Method introduced by Murio [Ref.8] and Guo, Murio and Roth [Ref.5]. The procedure allows for a direct discretization of the differential equation and it is generated by automatically filtering the noisy data by discrete mollification against a suitable averaging kernel and then using finite differences, marching in space, to numerically solve the associated well-posed problem. Once the temperature and the heat flux transient functions have been approximately recovered at the interface, it is a simple task to numerically identify the transient heat source if the nonlinear radiation law is known. On the other hand, if the surface is heated by a source at a rate proportional to a given function, we proceed to approximately recover the nonlinear surface radiation law describing the physical conditions at the interface, provided that the range of temperatures at the interface contain sufficient information.

In Section 2, we define the new identification problems with data specified on a

continuum of time and data errors measured in the  $L_2$  norm and derive rigorous stability bounds. The efficiency of the method is demonstrated in Section 3, where together with a description of the numerical procedure, we present the results of several computational experiments with rapidly varying and discontinuous profiles, for both linear - Newton cooling law - and nonlinear - Stefan-Boltzmann law - models. In all cases, numerical stability and good accuracy are achieved even for small time steps and high levels of noise in the data. Section 4 includes a summary and some conclusions.

## 2. Description of the Problem.

We consider a one-dimensional IHCP in a semi-infinite or finite slab, in which the temperature and heat flux histories  $f(t)$  and  $q(t)$  on the left-hand surface ( $x = 0$ ) are desired and unknown, and the temperature and heat flux at some interior point  $x = x_0$  or at the right-hand surface  $x = a$  are approximately measurable. Note that, equivalently, the data temperature histories might be measured at two interior points. For the semi-infinite medium,  $0 < x_0$  and for the finite slab,  $0 < x_0 \leq a$ . We assume linear heat conduction with constant coefficients and normalize the problem by dimensionless quantities. Without loss of generality, we consider  $x_0 = a = 1$  in all cases. The problem can be described mathematically as follows.

For the semi-infinite or finite slab, the unknown temperature  $u(x,t)$  satisfies respectively,

$$u_t(x,t) = u_{xx}(x,t), \quad t > 0, 0 < x < \infty \text{ or } 0 < x < 1, \quad (1a)$$

$$u(1,t) = F(t), \quad t > 0, \text{ with corresponding approximate data function } F_m(t), \quad (1b)$$

$$-u_x(1,t) = Q(t), \quad t > 0, \text{ with corresponding approximate data function } Q_m(t), \quad (1c)$$

$$u(x,0) = u_0(x), \quad 0 < x < \infty \text{ or } 0 < x < 1, \quad (1d)$$

$$u(0,t) = f(t), \quad t > 0, \text{ the desired but unknown temperature function,} \quad (1e)$$

$$\begin{aligned} -u_x(0,t) &= q(t) \\ &= E(u(0,t)) - g(t), \quad t > 0, \text{ the desired but unknown heat flux function.} \end{aligned} \quad (1f)$$

The nonlinear boundary condition (1f), indicates that the active surface radiates energy at a rate proportional to  $E$  and is heated at a rate proportional to the function  $g$ . Our aim is to obtain more detailed information about the boundary condition at the interface  $x = 0$ . More precisely, we want to estimate the function  $E$  if  $g$  is known or, reciprocally, we want to identify the source function  $g$  if the radiation law  $E$  is given.

We also assume that all the functions involved are  $L_2$  functions in any time interval of interest and use the corresponding  $L_2$  norm, as defined below, to measure errors:

$$\|f\| = \left[ \int_{t_1}^{t_2} |f(t)|^2 dt \right]^{1/2}.$$

In this setting, it is also natural to hypothesize that the exact data functions  $F(t)$  and  $Q(t)$  and the measured data functions  $F_m(t)$  and  $Q_m(t)$  satisfy the  $L_2$  data error bounds

$$\|F - F_m\| \leq \epsilon \quad \text{and} \quad \|Q - Q_m\| \leq \epsilon.$$

It is well known that solving for  $f(t)$  and  $q(t)$  from  $F(t)$  and  $Q(t)$  amplifies every Fourier frequency component of the error by the factor  $\exp[w/2]^{1/2}$ ,  $-\infty < w < \infty$ . This shows that the inverse problem is highly ill-posed in the high frequency components. See Murio [Ref.8] and Guo, Murio and Roth [Ref.5] for further discussions.

### Stabilized Problem.

The one-dimensional IHCP can be stabilized if instead of attempting to find the point values of the temperature function  $f(t)$  or the heat flux function  $q(t)$ , we attempt to reconstruct the  $\delta$ -mollification of the functions  $f$  and  $q$  at time  $t$ , given by

$$J_{\delta}f(t) = (\rho_{\delta} * f)(t), \quad J_{\delta}q(t) = (\rho_{\delta} * q)(t),$$

where

$$\rho_{\delta}(t) = \frac{1}{\delta \pi^{1/2}} \exp[-t^2/\delta^2]$$

is the one-dimensional Gaussian kernel of radius  $\delta > 0$ . The mollifier  $\rho_{\delta}(t)$  is always positive, falls to nearly zero outside the interval centered at the origin and radius  $3\delta$  and

$$(\rho_{\delta} * f)(t) = \int_{-\infty}^{\infty} \rho_{\delta}(\tau) f(t-\tau) d\tau$$

is the one-dimensional convolution of the functions  $\rho_{\delta}$  and  $f$ . We notice that  $J_{\delta}f(t)$  is a  $C^{\infty}$  (infinitely differentiable) function and that the mollifier has total integral 1. Mollifying system (1), we obtain the following associated problem: Attempt to find  $J_{\delta}f_m(t) = J_{\delta}u(0,t)$  and  $J_{\delta}q_m(t) = -J_{\delta}u_x(0,t)$  at some point  $t$  of interest and for some radius  $\delta > 0$ , given that  $J_{\delta}u(x,t)$  satisfies for the semi-infinite or finite slab respectively,

$$\begin{aligned} (J_{\delta}u)_t &= (J_{\delta}u)_{xx}, & t > 0, \quad 0 < x < \infty \text{ or } 0 < x < 1, \\ J_{\delta}u(1,t) &= J_{\delta}F_m(t), & t > 0, \\ -J_{\delta}u_x(1,t) &= J_{\delta}Q_m(t), & t > 0, \\ J_{\delta}u(x,0) &= J_{\delta}u_0(x,0), & 0 < x < \infty \text{ or } 0 < x < 1, \\ J_{\delta}u(0,t) &= J_{\delta}f_m(t), & t > 0, \text{ unknown,} \\ -J_{\delta}u_x(0,t) &= J_{\delta}q_m(t), & t > 0, \text{ unknown.} \end{aligned} \tag{2}$$

This problem and its solutions satisfy the following:

**Theorem 1.** Suppose that  $\|F - F_m\| \leq \epsilon$  and  $\|Q - Q_m\| \leq \epsilon$ . Then

- (i) Problem (2) is a formally stable problem with respect to perturbations in the data.
- (ii) If the exact boundary temperature function  $f(t)$  and the exact heat flux function  $q(t)$  have uniformly bounded first order derivatives on the bounded domain  $D = [0, T]$ , then  $J_{\delta}f_m$  and  $J_{\delta}q_m$  verify

$$\|f - J_{\delta} f_m\|_D \leq O(\delta) + 3\epsilon \exp[(2\delta)^{-2/3}] \quad (3)$$

and

$$\|q - J_{\delta} q_m\|_D \leq O(\delta) + \frac{\epsilon}{2} (1 + 3 \exp[\delta^{-2/3}]). \quad (4)$$

The proof of this statement can be found in Guo, Murio and Roth [Ref.5].

Once the mollified temperature and mollified heat flux functions have been evaluated at the interface, it is feasible to attempt to identify the source function  $g$  or the radiation energy function  $E$  given in formula (1f).

#### Identification of the source function $g$ .

Assuming that the radiation law at the active surface is known, according to (1f), the exact source function is given by

$$g(t) = E(f(t)) - q(t). \quad (5)$$

The approximate source function, denoted  $g_a(t)$ , is defined by

$$g_a(t) = E(J_{\delta} f_m(t)) - J_{\delta} q_m(t), \quad (6)$$

and in order to estimate the error, we suppose that the surface radiates energy at a rate proportional to  $[f(t)]^p$ . Here  $p$  is a positive integer, the value  $p = 1$  corresponding to Newton's law of cooling and  $p = 4$  to Stefan's radiation law.

The difference (5) - (6) gives

$$g(t) - g_a(t) = [f(t)]^p - [J_{\delta} f_m(t)]^p + q(t) - J_{\delta} q_m(t).$$

From the identity  $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \dots + ab^{n-2} + b^{n-1})$ , taking norms and introducing  $M = \max \{\|J_{\delta} f_m\|_{\infty, D}, \|f\|_{\infty, D}\}$ , we get

$$\|g - g_a\|_D \leq pM^{p-1} \|f - J_{\delta} f_m\|_D + \|q - J_{\delta} q_m\|_D.$$

Combining the last inequality with the upper bounds (3) and (4), we obtain the estimate

$$\|g - g_a\|_D \leq (pM^{p-1} + 1) \{O(\delta) + 3\epsilon \exp[\delta^{-2/3}]\}. \quad (7)$$

This shows that the identification of the source function  $g$  is stable with respect to errors in the data functions  $F$  and  $Q$ , for fixed  $p$  and  $\delta > 0$ .

#### Remarks:

1. Notice that the approximate source function  $g_a$  is actually a function of the radius of mollification  $\delta$ , the amount of noise in the data  $\epsilon$  and the exponent  $p$  in the radiation model  $E$ .
2. From a more theoretical point of view, inequality (7) can be used to show the convergence of  $g_a$  to  $g$  in the  $L_2$  norm. In fact, setting  $O(\delta) = C \delta$  for some constant  $C > 0$ , and choosing  $\delta = [\ln(1/\epsilon^{1/2})]^{-3/2}$ , after replacing these quantities in (7), we obtain

$$\|g - g_a\|_D \leq (pM^{p-1} + 1)(C[\ln(1/\epsilon^{1/2})]^{-3/2} + 3\epsilon^{1/2}).$$

This last inequality implies that, for the special selection of the radius of mollification indicated above,  $\|g - g_a\|_D \rightarrow 0$  as  $\epsilon \rightarrow 0$ , for any value of  $p$ .

#### Identification of the radiation law function $E$ .

From equation (1f) it follows that the exact function  $E$ , assuming that the

source function  $g$  is given, satisfies

$$E(u(0,t)) = E(f(t)) = g(t) + q(t). \quad (8)$$

The approximate function, denoted  $E_a$ , is defined by

$$E_a(J_\delta f_m(t)) = g(t) + J_\delta q_m(t). \quad (9)$$

Subtracting (8) from (9), taking norms and using inequality (4), we immediately have

$$\|E - E_a\|_D \leq O(\delta) + \frac{\epsilon}{2} (1 + 3 \exp[\delta^{-2/3}]). \quad (10)$$

This estimate also shows that the identification of the radiation law - as a function of time - is stable with respect to perturbations in the data functions  $F$  and  $Q$ , for a fixed  $\delta > 0$ , provided that the source function is known. However, this information is clearly not sufficient to identify the physical process at the interface. Nevertheless, since at each time  $t_i$  we know the ordered pairs  $(t_i, J_\delta f_m(t_i))$  and  $(t_i, E_a(t_i))$ , it is possible to collect the coordinates  $(J_\delta f_m(t_i), E_a(t_i))$  for  $t$  in a discrete subset of  $D$  and obtain a graph of the approximate functional relationship between the radiation law and the temperature at the interface. This is certainly always the case if the cardinality of the range of temperatures  $\{J_\delta f_m(t_i)\}$  is sufficiently large. Similar remarks to the ones in the previous paragraph, about the parameter dependency of  $E_a$  and convergence in the  $L_2$  norm of  $E_a$  to  $E$  as the quality of the data functions improve,  $\epsilon \rightarrow 0$ , also apply here.

The computational details are presented in the next section.

### 3. Numerical Procedure.

With  $v = J_\delta u$  and  $z = -\partial v / \partial x$ , system (2) is equivalent to

$$\begin{aligned} \frac{\partial v}{\partial t} &= -\frac{\partial z}{\partial x}, & t > 0, 0 < x < \infty \text{ or } 0 < x < 1, \\ z &= -\frac{\partial v}{\partial x}, & t > 0, 0 < x < \infty \text{ or } 0 < x < 1, \\ v(1,t) &= J_\delta F_m(t), & t > 0, \\ z(1,t) &= J_\delta Q_m(t), & t > 0, \\ v(x,0) &= J_\delta u_0(x,0), & 0 < x < \infty \text{ or } 0 < x < 1, \\ v(0,t) &= J_\delta f_m(t), & t > 0, \text{ unknown,} \\ z(0,t) &= J_\delta q_m(t), & t > 0, \text{ unknown.} \end{aligned} \quad (11)$$

Without loss of generality, we will seek to reconstruct the unknown mollified boundary temperature function  $J_\delta f_m$  and the mollified boundary heat flux function  $J_\delta q_m$  in the unit interval  $I = [0,1]$  of the time axis ( $x = 0$ ). Consider a uniform grid in the  $(x,t)$  space:  $\{(x_i = ih, t_n = nk), i = 0,1,\dots,N, Nh = 1; n = 0,1,\dots,M, Mk = L\}$ , where  $L$  depends on  $h$  and  $k$  in a way to be specified later,  $L > 1$ . Let the grid functions  $V$  and  $W$  be defined by

$$V_i^n = v(x_i, t_n), \quad W_i^n = z(x_i, t_n), \quad 0 \leq i \leq N, \quad 0 \leq n \leq M.$$

Notice that

$$V_N^n = J_\delta F_m(t_n), \quad W_N^n = J_\delta Q_m(t_n), \quad 0 \leq n \leq M,$$

and

$$V_i^0 = J_\delta u_0(x_i, 0), \quad 0 \leq i \leq N.$$

We approximate the partial differential equation in system (11) with the consistent finite difference schemes

$$\begin{aligned} W_{i-1}^n &= W_i^n - \frac{h}{2k}(V_i^{n+1} - V_i^{n-1}), \\ V_{i-1}^n &= V_i^n - h W_{i-1}^n, \\ i &= N, N-1, \dots, 1; \quad n = 1, 2, \dots, M-1. \end{aligned} \quad (12)$$

Notice that, as we march backward in the  $x$ -direction, we must drop the estimation of the interior temperature from the highest previous point in time. Since we want to evaluate  $\{V_0^n\}$  and  $\{W_0^n\}$  at the grid points of the unit time interval  $I = [0, 1]$  after  $N$  iterations, the minimum initial length  $L$  of the data sample interval in the time axis ( $x = 1$ ) needs to satisfy the condition  $L = kM = 1 - k + k/h$ .

Once the temperature  $J_\delta f_m$  and the heat flux  $J_\delta q_m$  have been reconstructed, we proceed with the approximate identification of the source function  $g_a$  or the radiation law function  $E_a$  as explained in Section 2.

#### Remarks:

1. The radius of mollification,  $\delta$ , can be selected automatically as a function of the level of noise in the data. In fact, for a given  $\epsilon > 0$ , there is a unique  $\delta > 0$ , such that

$$\|J_\delta F_m - F_m\|_D = \epsilon. \quad (13)$$

For the proof of this assertion and some discussions on the numerical implementation of this practical selection criterion, see Murio [Ref.9].

2. For the proof of the unconditional stability of the finite difference scheme (12) and the analysis of the convergence of the numerical solution of the mollified problem (11), the reader should consult Guo, Murio and Roth [Ref.5].

#### Numerical Results.

In order to test the accuracy and the stability properties of our method, in Problem 1, the approximate reconstruction of a source function  $g(t)$  and a nonlinear radiation law  $E(u(0, t))$  are investigated for a one-dimensional finite slab exposed to a heat flux data function at the free surface  $x = 1$  given by  $-u_x(1, t) = Q(t) = 0$ ,  $t > 0$ , and a temperature data function

$$u(1, t) = F(t) = \begin{cases} (t-0.2) - \frac{1}{6} - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \exp[-n^2 \pi^2 (t-0.2)], & t > 0.2, \\ 0, & 0 < t \leq 0.2. \end{cases}$$

The exact source solution to be approximately reconstructed at the interface  $x = 0$  has equation  $g(t) = E(u(0, t)) - q(t)$ , where  $E(u(0, t)) = [u(0, t)]^p$  and  $q(t) = -u_x(0, t)$ . We consider the values  $p = 1$  and  $p = 4$  corresponding to Newton's law of

cooling and Stefan's radiation law respectively. The exact radiation law at the interface is given by  $E(u(0,t)) = g(t) + q(t)$  and we only consider the nonlinear case  $p = 4$ . If the initial temperature distribution  $u(x,0)$  is zero, the exact temperature and heat flux functions at the interface are given respectively by a function at the free surface  $x = 1$  given by  $-u_x(1,t) = Q(t) = 0$ ,  $t > 0$ , and a temperature data function

$$u(0,t) = f(t) = \begin{cases} (t-0.2) + \frac{1}{3} - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \exp[-n^2 \pi^2 (t-0.2)], & t > 0.2, \\ 0, & 0 < t \leq 0.2. \end{cases}$$

and

$$-u_x(0,t) = q(t) = \begin{cases} 1, & t > 0.2, \\ 0, & 0 < t \leq 0.2. \end{cases}$$

With this information we generate the exact functions  $E(u(0,t))$  and  $g(t)$  for our model problem.

In Problem 2, we attempt to approximately reconstruct the transient source function  $g(t)$  for a semi-infinite body initially at zero temperature with data functions

$$u(1,t) = F(t) = \begin{cases} \operatorname{erfc}[(t-0.2)^{-1/2}/2], & t > 0.2, \\ 0, & 0 < t \leq 0.2, \end{cases}$$

and

$$-u_x(1,t) = Q(t) = \begin{cases} [\pi(t-0.2)]^{-1/2} \exp\{-[4](t-0.2)\}^{-1}, & t > 0.2, \\ 0, & 0 < t \leq 0.2. \end{cases}$$

The unique temperature solution at the interface is

$$u(0,t) = f(t) = \begin{cases} 1, & t > 0.2, \\ 0, & 0 < t \leq 0.2. \end{cases}$$

and the corresponding heat flux at the interface is

$$-u_x(0,t) = q(t) = \begin{cases} [\pi(t-0.2)]^{-1/2}, & t > 0.2, \\ 0, & 0 < t \leq 0.2. \end{cases}$$

In this case, we do not attempt the identification of the radiation law at the active boundary. The energy as a function of the interface temperature is either 0 or 1 for any value of  $p$  making its identification impossible. There is no enough information in the range of boundary temperatures which in this example is reduced to just two temperature values.

Since in practice only a discrete set of points is generally available, we shall assume that the data functions  $F_m$  and  $Q_m$  are discrete functions measured at equally spaced points in the time domain  $I = [0, L]$ , where  $L = 1 - k + k/h$ ,  $Nh = 1$ ,  $h = \Delta x$  and  $k = \Delta t$ . In order to compute  $J_{\delta} F_m(t_n)$  and  $J_{\delta} Q_m(t_n)$  in  $I$ , we need to extend the data functions in such a way that  $F_m$  and  $Q_m$  decay smoothly to zero in the interval  $I_{\delta_{\max}} = [-3\delta_{\max}, L+3\delta_{\max}]$  and both are zero in  $R - I_{\delta_{\max}}$ . In what follows, we consider the extended discrete data functions  $F_m$  and  $Q_m$  defined at equally spaced sample points on any interval of interest in the time axis.

The selection of the radius of mollification is implemented by solving the



discrete version of equation (13), using the bisection method.

Once the radii of mollification  $\delta_F$  and  $\delta_Q$ , associated with the data functions  $F_m$  and  $Q_m$  respectively, and the discrete filtered data functions  $J_\delta F_m(t_n) = V_N^n$  and  $J_\delta Q_m(t_n) = W_N^n$ ,  $0 \leq n \leq M$ , are determined with  $\delta = \max(\delta_F, \delta_Q)$ , we apply the finite difference algorithm described previously in this section, marching backward in the  $x$ -direction. The values  $V_0^n$  and  $W_0^n$ ,  $0 \leq n \leq M-N$ , so obtained, are then taken as the accepted approximations for the interface temperature and heat flux histories respectively at the different time locations at  $x = 0$ . Finally, we identify the approximate transient source function  $g_a$  or the approximate radiation law function  $E_a$  at the grid points of the time interval  $I = [0,1]$  using equations (6) and (9).

In all cases, we use  $h = \Delta x = 0.01$  and  $k = \Delta t = 0.01$ . Thus,  $N = 100$ ,  $L = 1.99$ ,  $M = 200$ ,  $\delta_{\max} = 0.1$  and  $I_{\delta_{\max}} = [-0.3, 2.29]$ . The noisy data is obtained by adding a random error to the exact data at every grid point  $t_n$  in  $I_{\delta_{\max}}$ :

$$\begin{aligned} F_m(t_n) &= F(t_n) + \epsilon_{n,1} \\ Q_m(t_n) &= Q(t_n) + \epsilon_{n,2}, \end{aligned}$$

where  $\epsilon_{n,1}$  and  $\epsilon_{n,2}$  are Gaussian variables of variance  $\sigma^2 = \epsilon^2$ .

If the discretized computed transient source function component is denoted by  $g_a^n$  and the true component is  $g^n = g(t_n)$ , we use the sample root mean square norm to measure the error in the discretized interval  $I = [0,1]$ . The solution error is then given by

$$\|g_a - g\|_I = \left[ \frac{1}{M-N} \sum_{n=1}^{M-N} (g_a^n - g^n)^2 \right]^{1/2}.$$

If the discretized computed radiation law function component is denoted by  $E_a^n = E_a(t_n)$  and the true component is  $E^n = E(t_n)$ , after evaluating the ordered pairs  $(V_0^n, E_a^n)$ ,  $0 \leq n \leq M-N$ , we obtain a graph of the approximate functional relationship between the radiation law and the temperature at the interface. This plot is then compared with the exact graph corresponding to the values  $(f(t_n), E(t_n))$  of the model problem.

Tables 1 and 2 show the results of our numerical experiments associated with Problems 1 and 2 respectively, when attempting to identify the transient source function at the interface. In all cases, the numerical stability of the method is confirmed. The uniformly smaller error norms in Problem 1 are expected since at time  $t = 0.2$  the exact source solution has a finite jump discontinuity while in Problem 2 the exact source solution has an infinite jump at time  $t = 0.2$ . For this reason, we have added an extra column in Table 2 indicating the error norms in the time interval  $[.3, 1]$ , after the discontinuity. It is clear that the method rapidly dissipates the effect of the singularity, a very desirable feature.

The qualitative behavior of the reconstructed transient source function for Problem 1 is illustrated in Figures 1 and 2 where the numerical solution for an average perturbation  $\epsilon = 0.005$  (full line) is plotted for  $p = 1$  (Newton's cooling law) and  $p = 4$  (Stefan's radiation law) respectively. In Figure 3 we show the graph associated with the reconstructed nonlinear radiation law as a function of the approximate temperature at the interface for  $p = 4$  (full line) and the exact

boundary radiation law (star symbols). Figures 4 and 5 show the computed source functions (full lines) for  $p = 1$  and  $p = 4$  respectively, for Problem 2 and for the noise level  $\epsilon = 0.005$ .

PROBLEM 1					
$p = 1$ (Newton)			$p = 4$ (Stefan)		
$\epsilon$	$\delta$	Error norm	$\epsilon$	$\delta$	Error norm
0.000	0.04	0.0866	0.000	0.04	0.0867
0.002	0.06	0.0921	0.002	0.06	0.0929
0.005	0.06	0.1014	0.005	0.06	0.1038

Table 1. Error norm as a function of the level of noise

PROBLEM 2					
$p = 1$ (Newton)			$p = 4$ (Stefan)		
$\epsilon$	$\delta$	Error norm [0,1]/[.3,1]	$\epsilon$	$\delta$	Error norm [0,1]/[.3,1]
0.000	0.04	.5208/.0183	0.000	0.04	.5135/.0373
0.002	0.06	.5560/.0631	0.002	0.06	.5673/.0675
0.005	0.06	.5879/.1108	0.005	0.06	.5935/.1375

Table 2. Error norm as a function of the level of noise

#### 4. Conclusions.

An explicit and unconditionally stable space marching finite difference method for the solution of the one-dimensional transient inverse heat conduction problem has been implemented for the numerical identification of surface heat sources, if the energy radiation law at the active interface is known, and to the numerical identification of the nonlinear surface radiation law if the surface is heated by a source at a rate proportional to a given function and the interface temperature contains enough information.

The computational procedure is applied to two examples corresponding to Newton cooling law and to Stefan-Boltzmann radiation law. In both problems, the source functions to be identified have discontinuous histories and in one case an infinite jump. The algorithm restores stability with respect to the data, which is essential for the introduction of the inverse problem approach, and good accuracy is obtained, even for small time sample intervals and relative high noise levels in the data.

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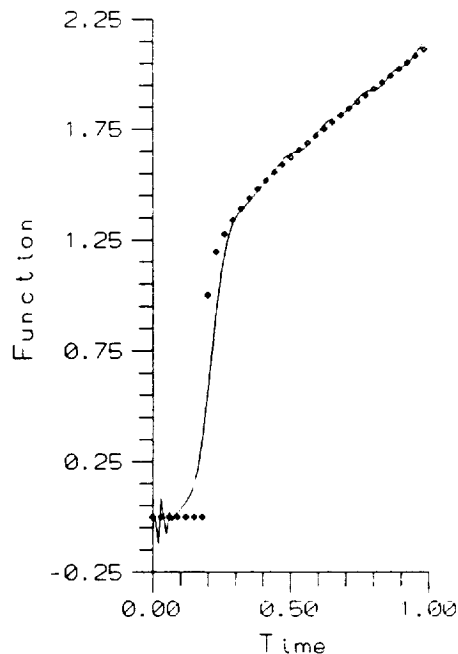


Fig.1 Source function for Newton Law  
Problem 1,  $\varepsilon=0.005$ ,  $\delta=0.06$ ,  $\Delta t=0.01$   
Exact: (\* \* \*) ; Computed: (—)

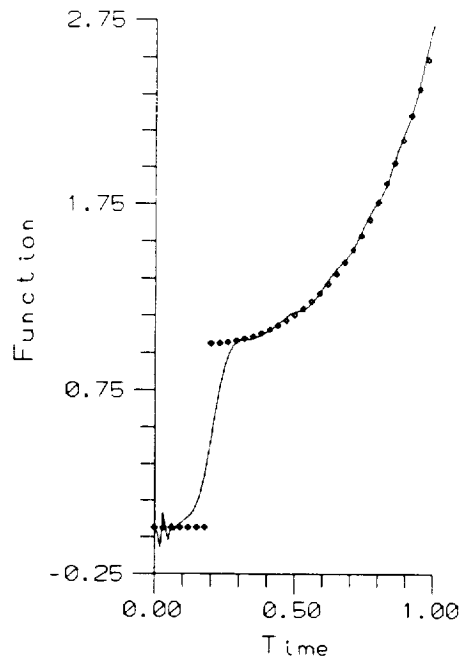


Fig.2 Source function for Stefan Law  
Problem 1,  $\epsilon=0.005$ ,  $\delta=0.06$ ,  $\Delta t=0.01$   
Exact: (\* \* \*) ; Computed: (—)

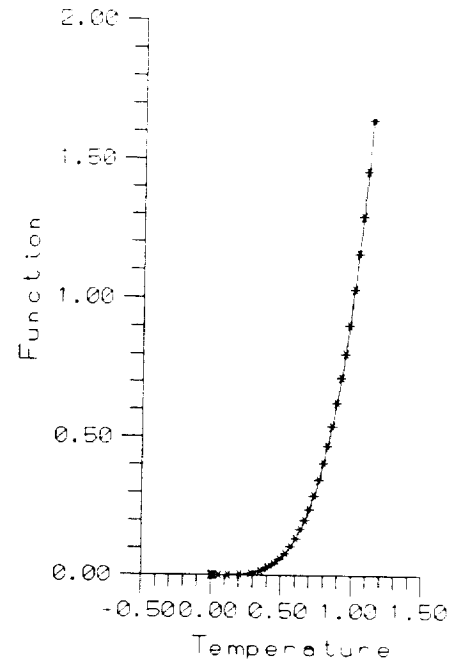


Fig.3 Reconstructed Stefan radiation Law  
Problem 1,  $\epsilon=0.005$ ,  $\delta=0.06$ ,  $\Delta t=0.01$   
Exact: (\* \* \*) ; Computed: (—)

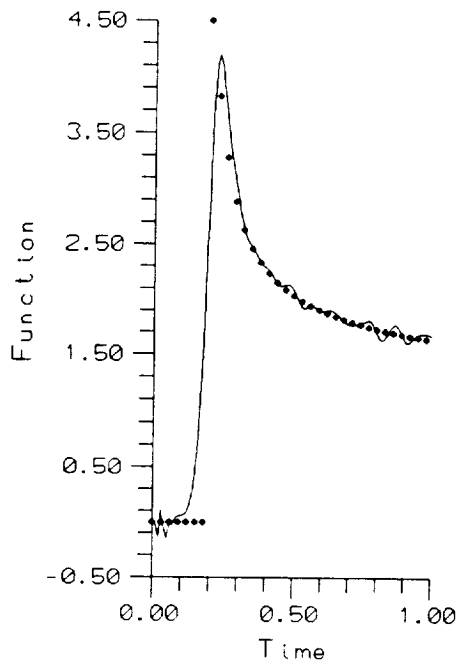


Fig.4 Source function for Newton Law  
Problem 2,  $\epsilon=0.005$ ,  $\delta=0.06$ ,  $\Delta t=0.01$   
Exact: (\* \* \*) ; Computed: (—)

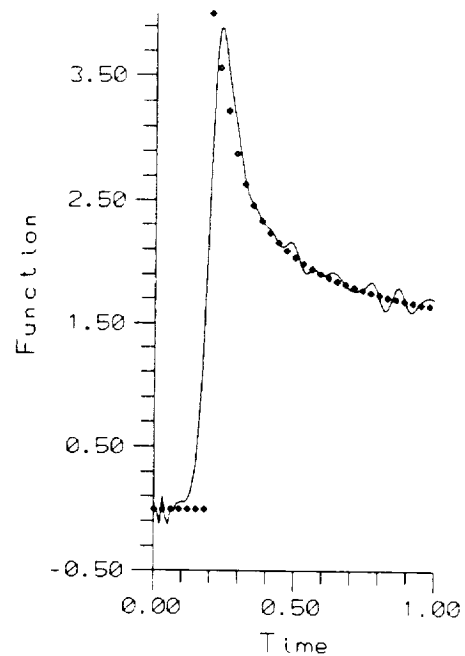


Fig.5 Source function for Stefan Law  
Problem 2,  $\epsilon=0.005$ ,  $\delta=0.06$ ,  $\Delta t=0.01$   
Exact: (\* \* \*) ; Computed: (—)